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RELAXATION OF THE CONTACT SHEARING STRESS IN PROBLEMS WITH SLIDING FRICTION†

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A number of formulations of the contact problem of the theory of elasticity when there is friction present in the case of unidirectional relative sliding of the interacting bodies (movable coupling of the sliding guide type) are considered. Particular attention is given to the behaviour with time of the contact shearing stress q_1 in a plane perpendicular to the sliding direction. It is established that this stress relaxes (decays) with time, which may serve as a basis for formulations of contact problems with unidirectional sliding of bodies, assuming that there is no stress q_1 in the contact area [1–4].

1. THE ONE-DIMENSIONAL PROBLEM

We will consider, as a simple example of contact interaction with unidirectional sliding of bodies, a system of two bodies, similarly loaded and connected by a deformed spring, which slides with friction along a flat surface with a constant velocity V in a direction perpendicular to the axis of the spring (Fig. 1). This will enable us to investigate the characteristic features of such interaction. The value of the friction force $\mathbf{F}$ acting on each body is given by Coulomb's law: $|\mathbf{F}| \equiv F = fQ + F_a$, where $0 \le f$ is the friction coefficient, Q is the load on the body and F_a is the adhesion interaction force; in this case $\mathbf{F} = -F\mathbf{V}_s/V_s$, where $\mathbf{V}_s$ is the sliding velocity of the body. As regards the value of T (the elasticity force of the spring) it is assumed that it is linearly related to the deformation (Hooke's law): $T = -\gamma(x - \bar{x})$, where γ is the stiffness of the spring, x is the distance between the bodies and $\bar{x}$ is the value of x for the underformed spring.

When the elastic system considered slides, the distance x changes. If we set up the balance of the forces, equating the projection of the friction force $\mathbf{F}$ onto the axis of the spring to the value T of its force of elasticity, we arrive at the following equation for $x(t)$

$$\frac{F dx(t)/dt}{\sqrt{(dx(t)/dt)^2 + V^2}} + \gamma(x(t) - \bar{x}) = 0 \tag{1.1}$$

We will further assume that $t \ge 0$ and specify the initial condition $x(0) = x_0$.

The implicit expression for the solution of Eq. (1.1) has the form

$$H(\psi(t)) - H(\psi_0) = -\omega t$$
$$H(\xi) = \frac{1}{2}\ln\frac{1-\sqrt{1-\xi^2}}{1+\sqrt{1-\xi^2}} + \sqrt{1-\xi^2}, \quad \psi(t) = \gamma\frac{x(t)-\bar{x}}{F}, \quad \psi_0 = \gamma\frac{x_0 - \bar{x}}{F}, \quad \omega = \gamma\frac{V}{F} \tag{1.2}$$

By virtue of Eq. (1.2) $H(\psi(t)) \to -\infty$ when $t \to \infty$, whence it follows that $\psi(t) \to 0$ when $t \to -\infty$, since the function $H(\xi)$ is even, increases monotonically when $\xi > 0$ and $H(\xi) \to -\infty$ when $\xi \to 0 + 0$. The limit relation obtained for $\psi(t)$ denotes that $T(t) = -F\psi(t) \to 0$ when $t \to \infty$, i.e. in the elastic system

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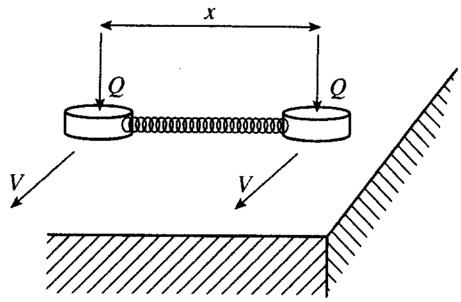

Fig. 1

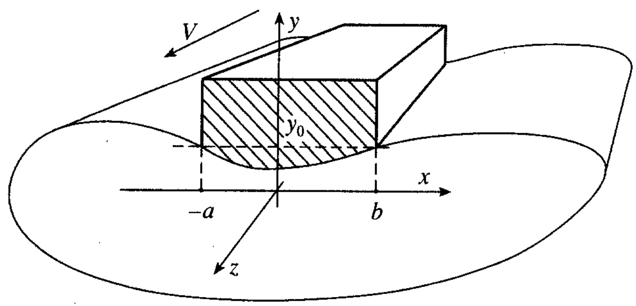

Fig. 2

considered there is relaxation of the force of elasticity of the spring tangential to the sliding surface.

When $|\gamma F^{-1}(x_0 - \bar{x})| \ll 1$, from relation (1.2) we can obtain explicit expressions for $x(t)$ and $T(x)$

$$x(t) - \bar{x} = (x_0 - \bar{x})e^{-\omega t}, \quad T(t) = T_0 e^{-\omega t} \tag{1.3}$$

2. BASIC RELATIONS FOR AN ELASTIC SOLID

Consider a cylindrical elastic solid, bounded by the surface Γ. Following the well-known approach [5], we will connect with a certain point of this body the origin of a system of coordinates x, y, z, directing the z axis along the generatrix of the boundary of Γ (Fig. 2). We will denote by u, υ, w and q_1, q_2, q_3 the components of the vectors of the boundary displacement and the boundary stress in the system x, y, z.

We will assume that the boundary Γ has a plane section, situated parallel to the coordinate plane xz with ordinate $y = y_0$, and unidirectional sliding of a cylindrical punch occurs along this part in the direction of the z axis with velocity V (Fig. 2). There is no displacement of the punch along the x axis, and the dimensions a and b of its contact region with the solid are assumed to be constant. Moreover, the specific load $Q > 0$ on the punch along the y axis is assumed to be constant, in which case we have the following equilibrium condition

$$Q = -\int_{-a}^{b} q_2(x, t)dx \tag{2.1}$$

The interaction of the punch with the elastic solid described corresponds to a mixed boundary-value problem of the theory of elasticity. The boundary conditions outside the contact region can have a different form depending on the possible loading and clamping conditions of the corresponding parts of the boundary [1, 6], whereas in the contact region these conditions are determined by Coulomb's friction law and the contact condition. Unlike the integral form of Coulomb's law used in Section 1, we will present it here in the form

$$\mathbf{q}_\tau = \frac{\mathbf{V}_s}{V_s}(-fq_2 + \tau_a) \tag{2.2}$$

where $\mathbf{q}_\tau$ is the shear stress vector at a point of the boundary of the solid, $\mathbf{V}_s$ is the velocity of sliding of the punch with respect to this point, and $0 \le \tau_a$ is the adhesion component of the friction. Taking into account the fact that $\mathbf{q}_\tau = (q_1, 0, q_3)$ and $\mathbf{V}_s = (-\dot{u}, 0, V-\dot{w})$, the vector equality (2.2) can be written in the form of its components

$$q_1(x,t) = -\frac{\dot{u}(x,t)}{V_s(x,t)}\tau(x,t), \quad q_3(x,t) = \frac{V-\dot{w}(x,t)}{V_s(x,t)}\tau(x,t), \quad x \in [-a,b] \tag{2.3}$$

where

$$\tau(x,t) \equiv -fq_2(x,t) + \tau_a, \quad V_s(x,t) = \sqrt{\dot{u}^2(x,t) + (V-\dot{w}(x,t))^2} \tag{2.4}$$

The dot above a symbol denotes a partial derivative with respect to time t, $t \ge 0$.

Note that the functions $q_i(x, t)$ $(i = 1, 2, 3)$ which satisfy Eqs (2.3) are related by the equation

$$q_1^2(x,t) + q_3^2(x,t) = \tau^2(x,t) \equiv [-fq_2(x,t) + \tau_a]^2 \tag{2.5}$$

which follows directly from relations (2.3) and (2.4).

The condition for the punch to be in contact with the elastic solid has the form

$$\upsilon(x,t) = g(x) - \delta(t), \quad x \in [-a,b] \tag{2.6}$$

where $\delta(t) = -\upsilon(0, t)$ is the value of the sagging of the boundary of the body when $x = 0$, $g(x)$ is a function describing the shape of the punch and $g(0) = 0$.

Equations (2.3) describe the rate of variation with time of the displacements u and w within the contact region as a function of the contact stresses q_i, which, together with the equilibrium condition (2.1) and the contact condition (2.6) as well as the boundary conditions outside the contact region, defines the kinetics of the variation with time of the stress–strain state of the elastic solid. We will further assume that the following initial conditions are given

$$q_i(x,0) = q_{i0}(x), \quad i = 1,2,3 \tag{2.7}$$

In the case of a high sliding velocity V of the punch, when

$$|\dot{u}(x,t)/V| \sim \varepsilon \ll 1, \quad |\dot{w}(x,t)/V| \sim \varepsilon \ll 1 \tag{2.8}$$

the following expressions hold for the fractions occurring on the right-hand sides of Eqs (2.3) with an accuracy of $O(\varepsilon^2)$

$$\dot{u}(x,t)/V_s(x,t) = \dot{u}(x,t)/V, \quad (V-\dot{w}(x,t))/V_s(x,t) = 1$$

using which, Eqs (2.3) can be written in the form

$$q_1(x,t) = -\dot{u}(x,t)\tau(x,t)/V, \quad q_3(x,t) = \tau(x,t), \quad x \in [-a,b] \tag{2.9}$$

We can draw a number of conclusions regarding the structure of the stress–strain state of an elastic solid under conditions of unidirectional sliding, which will enable us later to reduce the solution of the corresponding three-dimensional problem to a simpler two-dimensional problem. We will denote by u_i and σ_{ij} the components of the displacement vector and the stress tensor, ascribing the subscripts i, $j = 1, 2, 3$ to the coordinate axes x, y, z respectively. For the interaction of the punch with the elastic solid considered, the stress–strain state of the elastic solid, as well as its geometry, do not change along the z axis, and hence the derivatives of the displacements u_i with respect to z are equal to zero. If we take this fact into account, and, following the well-known approach in [1], write the differential equations of elastic equilibrium of the solid in terms of displacements (Lamé's equations) and Hooke's law, it turns out that some of these equations will describe plane deformation [7] and contain the components $u_1, u_2, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}$, whereas the remaining equalities describe antiplane deformation (pure shear) [6] and contain the components $u_3, \sigma_{13}, \sigma_{23}$. With these systems of the stress–strain states, in turn, proved to be connected the different sets of components u, υ, q_1, q_2 and w, q_3:

for plane deformation

$$u_1|_\Gamma = u, \quad u_2|_\Gamma = \upsilon, \quad (\sigma_{11}n_1 + \sigma_{12}n_2)|_\Gamma = q_1, \quad (\sigma_{21}n_1 + \sigma_{22}n_2)|_\Gamma = q_2 \tag{2.10}$$

for antiplane deformation

$$u_3|_\Gamma = w, \quad (\sigma_{31}n_1 + \sigma_{32}n_2)|_\Gamma = q_3 \tag{2.11}$$

where $n_1, n_2, n_3 = 0$ are the components of the vector of the outward normal to the boundary Γ of the solid.

In general, when we have Eqs (2.3), both stress–strain state systems turn out to be mutually connected via the boundary conditions (Eqs (2.10) and (2.11)). In fact, the first equation of (2.3) contains, in addition to u, q_1 and q_2, a component w, connected with the antiplane deformation system, whereas the second equation of (2.3) contains, in addition to w and q_3, the components u and q_2, connected with the plane deformation system. The position is simplified if we used Eqs (2.9), since the first equation of (2.9) only contains the components u, q_1 and q_2, connected, by virtue of relations (2.10), with plane deformation. Together with the equilibrium condition (2.1) and the contact condition (2.6), this equation forms a set of boundary conditions for the plane-deformation equations, after solving which and determining q_2 the second equation of (2.9) can be used as the boundary condition for the antiplane-deformation equations.

Note that it was pointed out earlier in [3] that antiplane deformation has no effect on plane deformation. A similar situation occurs when the axisymmetric contact problem with friction is considered [1].

3. A THIN LAYER (A WINKLER BODY)

Suppose a thin layer of thickness h, connected with an absolutely rigid base in its lower boundary, serves as the elastic body (Fig. 3). We also connect the origin of a system of coordinates with a point of this boundary. Assuming the layer is thin, i.e. when $h \ll (a + b)$, its deformation is described by the model of a Winkler body [8]

$$u = \alpha q_1, \quad v = \beta q_2, \quad w = \alpha q_3; \; \alpha = \frac{h}{G}, \quad \beta = \frac{(1-2\nu)h}{2G(1-\nu)} \tag{3.1}$$

where G is the shear modulus and ν is Poisson's ratio.

Replacing the variable v in the second relation of (3.1) by the right-hand side of the contact condition (2.6) we obtain the expression

$$q_2(x, t) = \beta^{-1}[g(x) - \delta(t)] \le 0 \tag{3.2}$$

and substituting this into the equilibrium condition (2.1) and taking into account the fact that Q, a and b are constants, we can establish that the quantity δ occurring in the contact condition is independent of time

$$\delta(t) = (a+b)^{-1}\left[\beta Q + \int_{-a}^{b} g(x)dx\right] = \text{const}$$

Bearing this result and expression (3.2) for $q_2(x, t)$ in mind, we can convert Eq. (2.4), which defines $\tau(x, t)$, as follows:

$$\tau(x, t) = -f\beta^{-1}[g(x) - \delta] + \tau_a \equiv \tau_0(x) \ge 0 \tag{3.3}$$

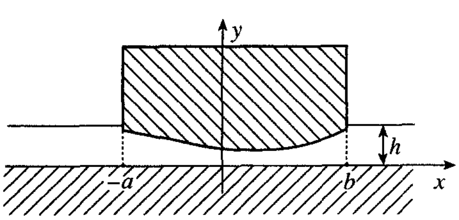

Fig. 3

We now use Eqs (2.3). Substituting the first and third expressions of (3.1) into them and taking expression (3.3) into account, we obtain

$$q_1(x,t) = -\alpha\frac{\dot{q}_1(x,t)}{V_s(x,t)}\tau_0(x), \quad q_3(x,t) = \frac{V-\alpha\dot{q}_3(x,t)}{V_s(x,t)}\tau_0(x), \quad x\in[-a,b] \tag{3.4}$$

It follows directly from the first equation of (3.4), in view of the fact that α, τ_0 and V_s are non-negative, that $\dot{q}_1(x,t) > 0$ when $q_1(x,t) < 0$ and $\dot{q}_1(x,t) < 0$ when $q_1(x,t) > 0$, i.e. the function which satisfies relations (3.4) decays with time

$$|q_1(x,t)| \to 0 \quad \text{as} \quad t\to\infty, \quad x\in[-a,b] \tag{3.5}$$

Note that by the conclusions reached in Section 2 regarding the interaction of plane and antiplane deformation systems when the general equations (2.3) are used, each of equalities (3.4) contains quantities which refer to both of these systems and each of the equalities depends on the other. Nevertheless, for Eqs (3.4) we can obtain an exact solution if we convert them as follows: divide the first equation by the second and then eliminate the derivatives $\dot{q}_1(x,t)$ and $\dot{q}_3(x,t)$ in turn from the equation obtained using Eqs (2.5) and (3.3). As a result, Eqs (3.4) take the form

$$\dot{q}_1(x,t) = -\frac{V}{\alpha}\frac{q_1(x,t)q_3(x,t)}{\tau_0^2(x)}, \quad \dot{q}_3(x,t) = \frac{V}{\alpha}\left[1-\frac{q_3^2(x,t)}{\tau_0^2(x)}\right], \quad x\in[-a,b] \tag{3.6}$$

Solving the second equation of (3.6) first and then solving the first equation of (3.6) using the function $q_3(x,t)$ obtained, taking the initial conditions (2.7) into account, we obtain the expressions

$$q_1(x,t) = \frac{2q_{10}(x)^{-\omega(x)t}}{B_+(x,t)}, \quad q_3(x,t) = \tau_0(x)\frac{B_-(x,t)}{B_+(x,t)} \tag{3.7}$$

where

$$B_\pm(x,t) = A_+(x)\pm A_-(x)e^{-2\omega(x)t}, \quad A_\pm(x) = 1\pm\frac{q_{30}(x)}{\tau_0(x)}, \quad \omega(x) = \frac{V}{\alpha\tau_0(x)}$$

If we use Eqs (2.9) instead of (2.3) we have

$$q_1(x,t) = q_{10}(x)e^{-\omega(x)t} \tag{3.8}$$

As might have been expected, both expressions (3.7) and (3.8) satisfy relation (3.5), in which case the relaxation (decay) of the stress $q_1(x,t)$ with time occurs exponentially.

Remark 1. The initial shear stress q_1 in the case considered may be due to a preliminary shift of the punch along the x axis. For example, if this shift leads to complete slippage of the punch, then, by virtue of relations (2.2), (2.5) and (3.3), $q_{10}(x) = \pm\tau_0(x)$, $q_{30}(x) = 0$ and expressions (3.7) take the form

$$q_1(x,t) = \pm\tau_0(x)/\mathrm{ch}(\omega(x)t), \quad q_3(x,t) = \tau_0(x)\mathrm{th}(\omega(x)t) \tag{3.9}$$

4. A COMPOSITION CONSISTING OF A THIN LAYER AND A HALF-SPACE

Unlike the previous formulation, we assume here that the base connected with the layer behaves as an elastic half-space, while the punch itself and its loading conditions are symmetrical about the y axis ($b = a$), in which case the origin of coordinates is assumed to be connected with a point on the upper boundary of the layer, situated in the middle of the contact region $[-a, a]$. With this choice of the system of coordinates, the contact condition (2.6) takes the form

$$v(x,t) = g(x), \quad x\in[-a,a] \tag{4.1}$$

where, by virtue of the symmetry of the problem, $g(x)$ is an even function. Moreover, the symmetry of the problem enables us to assume that $q_1(x, t)$ is an odd function of x while $q_2(x, t)$ is an even function of x.

In addition to the previous condition that the layer is thin ($h \ll 2a$), we will assume that $n \equiv G_1/G_2 \ll 1$, i.e. the layer is soft compared with half-space. Here and henceforth the subscripts 1 and 2 are assigned to quantities belonging to the layer and half-space respectively. The purpose of our further calculations, as previously, will be to analyse the behaviour with time of the contact shear stress q_1.

It was shown above (Section 2), that under conditions of unidirectional sliding of the punch along an elastic body of cylindrical form, the stress–strain state of the latter splits into two systems: plane and antiplane deformation. It is easy to establish that in the two-layer composition considered here, the same two systems of stress–strain states occur, where the first of these is determined, according to relations (2.10), by the components u, υ, q_1 and q_2 on the upper boundary of the layer (for $y = 0$). In fact, since there is no change in the loading conditions and the geometry of the layer and the half-space along the z axis, the stress–strain state of each of them splits into the two systems indicated above (Section 2), and any change in the components w and q_3 on the upper boundary of the layer, when u, υ, q_1 and q_2 are unchanged, has no effect on the plane deformation of the layer (see Eqs (2.10) and (2.11)) and, in particular, on the values of the components u, υ, q_1 and q_2 when $y = -h$ (i.e. at the interface of the layer and the half-space), by which the plane-deformed state of the half-spaces as a whole is determined.

The presence of a plane-deformation system, including the stress q_1, in the elastic body considered, enables us to use the plane-deformation equations for the strip – half-plane composition for the purpose of determining the stress q_1. According to these equations, when $h \ll 2a$ and $n \ll 1$ we have the following relations between the components u, υ, q_1 and q_2 on the upper boundary of the layer [9]

$$\begin{aligned} Gu'(x,t) &= A_\tau q_1'(x,t) + nM_c q_2(x,t) + nM_D(\mathcal{H}q_1)(x,t) \\ G\upsilon'(x,t) &= A_\nu q_2'(x,t) - nM_c q_1(x,t) + nM_D(\mathcal{H}q_2)(x,t) \end{aligned} \tag{4.2}$$

where

$$G = G_1, \quad M_C = \frac{1-2\nu_2}{2}, \quad M_D = \frac{1-\nu_2}{\pi}$$

$$A_\tau = [1+(1-2\nu_2)n]h + O(n^2), \quad A_\nu = \frac{1-2\nu_1}{2(1-\nu_1)}\left[1 - \frac{2\nu_1(1-2\nu_2)}{1-2\nu_1}n\right]h + O(n^2) \tag{4.3}$$

$$(\mathcal{H}\varphi)(x) = \int_{-a}^{a} \frac{\varphi(\xi)d\xi}{\xi - x}$$

The prime denotes differentiation with respect to x.

We will convert Eqs (4.2). To do this we will assume that the functions $q_{1,2}(x, t)$ in the contact area are quadratically summable with respect x:

$$q_{1,2}(x,t) \in L_2[-a,a]; \quad t \in [0,T] \tag{4.4}$$

where $0 < T$ is a certain quantity, where here and henceforth the integrals are understood in the Lebesgue sense [10]. The stresses $q_i(x, t)$ ($i = 1, 2, 3$) are assumed to be equal to zero outside the contact area.

We introduce the following operators

$$(\mathcal{L}_0\varphi)(x) = -\int_{-a}^{a} \varphi(\xi)\ln\left|\frac{\xi - x}{\xi}\right| d\xi, \quad (\mathcal{P}_0\varphi)(x) = \int_0^x \varphi(\xi)d\xi, \quad \varphi(x) \in L_2[-a,a] \tag{4.5}$$

and note the equality which connects the operator $\mathcal{H}$ of the form (4.3) with $\mathcal{L}_0$ [11]

$$\int_0^x (\mathcal{H}\varphi)(\xi)d\xi = (\mathcal{L}_0\varphi)(x), \quad \varphi(x) \in L_2[-a,a] \tag{4.6}$$

Returning to equalities (4.2), we replace the variable x by ξ in them and integrate the result with respect to ξ from 0 to x. Using relations (4.4)–(4.6), we arrive at the equalities

$$\begin{aligned} A_\tau q_1(x,t) &= Gu(x,t) - nM_C(\mathcal{P}_0 q_2)(x,t) - nM_D(\mathcal{L}_0 q_1)(x,t) \\ A_\nu[q_2(x,t) - q_2(0,t)] &= G\upsilon(x,t) + nM_C(\mathcal{P}_0 q_1)(x,t) - nM_D(\mathcal{L}_0 q_2)(x,t) \end{aligned} \tag{4.7}$$

By acting on the right-hand sides of equalities (4.7) with the operator $\mathcal{P}_0$ we can obtain expressions for $(\mathcal{P}_0 q_{1,2})(x,t)$, and substituting these into the equations we obtain

$$\begin{aligned} &A_\tau q_1(x,t) = Gu(x,t) - nM_D(\mathcal{L}_0 q_1)(x,t) - \\ &- nM_C q_2(0,t)x - nM_C\frac{G}{A_\nu}(\mathcal{P}_0\upsilon)(x,t) - n^2 A_\tau \rho_1(x,t) \\ &A_\nu[q_2(x,t) - q_2(0,t)] = G\upsilon(x,t) - nM_D(\mathcal{L}_0 q_2)(x,t) + nM_C\frac{G}{A_\tau}(\mathcal{P}_0 u)(x,t) - n^2 A_\nu \rho_2(x,t) \end{aligned} \tag{4.8}$$

where the following limit holds for the functions $\rho_{1,2}(x,t)$

$$|\rho_{1,2}(x,t)| \le 2a^{3/2}|A_\tau A_\nu|^{-1}\max\{\|q_1\|, \|q_2\|\}$$

which shows that the last terms in Eqs (4.8) are of the order of n^2 and can be omitted when $n \ll 1$. Here and henceforth $\|\varphi\|$ is the norm of the function $\varphi(x)$ in the space $L_2[-a, a]$.

To analyse the behaviour of the shear stress $q_1(x,t)$ we will use the first equality of (4.8), which, in addition to the required function $q_1(x,t)$, contains the unknown functions $q_2(0,t)$ and $\upsilon(x,t)$. However, the latter can easily be eliminated from this equation. In fact, the function $\upsilon(x,t)$ is expressed in terms of $g(x)$ by the contact condition (4.1), while the following expression holds for the function $q_2(0,t)$

$$q_2(0,t) = -(2aA_\nu)^{-1}\left[A_\nu Q + \int_{-a}^{a} g(x)dx\right] + O(n)$$

which is obtained if we integrate the second equality of (4.8) with respect to x from $-a$ to a and take into account the equilibrium condition (2.1) and the contact condition (4.1). If these expressions are taken into account the first equality of (4.8) takes the form

$$A_\tau q_1(x,t) = Gu(x,t) - nM_D(\mathcal{L}_0 q_1)(x,t) + U(x) \tag{4.9}$$

where $U(x)$ is a known function which depends on Q and $g(x)$.

We will further consider the case (2.8) for a high sliding velocity V of the punch when its frictional interaction with the layer has a purely adhesion form, i.e. $f = 0$. These assumptions enable us to use the first equality of (2.9) to describe the kinetics of the variation of the stress q_1 by representing it, taking the first expression of (2.4) into account, in the form $q_1(x,t) = -\tau_a V^{-1}\dot{u}(x,t)$, or, after integrating with respect to t,

$$u(x,t) = -\frac{V}{\tau_a}\int_0^t q_1(x,\tau)d\tau + u(x,0) \tag{4.10}$$

If we replace the function $u(x,t)$ in Eq. (4.9) by the right-hand side of (4.10) and carry out a simple reduction of this equation, using the fact that $q_1(x,t)$ is odd in x, and the definition of the function $U(x)$ in terms of the initial distribution $q_{10}(x) = q_1(x, 0)$, we can finally obtain the following equation for $q_1(x,t)$

$$A_0 q_1(x,t) = -\int_0^t q_1(x,\tau)d\tau - A_1(\mathcal{L}q_1)(x,t) + A_0 q_{10}(x) + A_1(\mathcal{L}q_{10})(x) \tag{4.11}$$

where

$$(\mathcal{L}\varphi)(x) = -\int_{-a}^{a}\varphi(\xi)\ln\left|\frac{\xi - x}{a}\right|d\xi; \quad A_0 = \frac{\tau_a A_\tau}{VG}, \quad A_1 = \frac{\tau_a n M_D}{VG} \tag{4.12}$$

and $A_{0,1} > 0$ for $n \ll 1$ and $\nu \in [0, 1/2)$.

Equation (4.11) contains no quantities relating to the antiplane deformation system of the body, and this agrees with the conclusion reached in Section 2 that the plane deformation is independent of the antiplane deformation when using Eqs (2.9).

Proceeding to the solution of Eq. (4.11), we note that the operator $\mathcal{L}$ is strictly positive [2], self-conjugate (since the kernel $\ln|(\xi - x)/a$ is symmetrical) and is completely continuous in the space $L_2[-a, a]$ [12] where the latter is Hilbert-separable. These properties ensure the existence of systems of eigenvalues λ_k and functions $X_k(x)$ of the operator $\mathcal{L}$ [12]

$$\lambda_k(\mathcal{L}X_k)(x) = X_k(x), \quad k = 1, 2, \ldots \tag{4.13}$$

where the system $\{X_k(x)\}$, being complete in the space $L_2[-a, a]$, forms an orthonormalized basis in it, such, that for any function $\varphi(x)$ from this space (everywhere henceforth, unless otherwise stated, the summation is carried out from $k = 1$ to $k = \infty$)

$$(\mathcal{L}\varphi)(x) = \sum \lambda_k^{-1}(\varphi, X_k)X_k(x); \quad (\varphi, X_k) = \int_{-a}^{a} \varphi(x)X_k(x)dx \tag{4.14}$$

From Eq. (4.13), provided that $\mathcal{L}$ is a strictly positive operator, it follows that the quantities λ_k are positive. Henceforth λ_k will be numbered in increasing order:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$$

Using assumption (4.4) and following Fourier's method [13, 14], we can represent the function $q_1(x, t)$ in the form of an expansion in an orthonormalized basis $\{X_k(x)\}$

$$q_1(x, t) = \sum a_k(t)X_k(x), \quad t \in [0, T] \tag{4.15}$$

As is well known [13, 14], to determine the coefficients $a_k(t)$ of this expansion we must substitute series (4.15) into Eq. (4.11) and formally introduce the operation of integration into it under the sign of the sum of the series, after which, using Eq. (4.13) and the fact that the system $\{X_k(x)\}$ is orthonormalized, we can obtain an ordinary differential equations for $a_k(t)$ with the solution

$$a_k(t) = a_k(0)e^{-\omega_k t}, \quad \omega_k = \lambda_k(A_0\lambda_k + A_1)^{-1} > 0 \tag{4.16}$$

Taking the initial conditions (2.7) and expansion (4.15) into account, the values $a_k(0)$ will be found from the equality

$$q_{10}(x) = \sum a_k(0)X_k(x) \tag{4.17}$$

Then, assuming, relative to the unknown function $q_{10}(x)$, that

$$q_{10}(x) \in L_2[-a, a] \tag{4.18}$$

we obtain: $a_k(0) = (q_{10}, X_k)$, where, by Parseval's equality [12]

$$\|q_{10}\|^2 \equiv \int_{-a}^{a} q_{10}^2(x)dx = \sum a_k^2(0) < \infty \tag{4.19}$$

Before establishing the conditions under which series (4.15) with coefficients (4.16) can be used as a solution of the problem in question, we will note some results connected with the convergence of this series, uniform in $t \in [0, T]$.

Assertion 1. Suppose series (4.17) converges in the usual sense for a certain $x \in [-a, a]$. Then, for this x we have the convergence of series (4.15) with coefficients (4.16), uniform in $t \in [0, T]$, and the equation

$$\int_0^t [\sum a_k(0)e^{-\omega_k \tau}X_k(x)]d\tau = \sum\left[\int_0^t e^{-\omega_k \tau}d\tau\right]a_k(0)X_k(x) \tag{4.20}$$

Proof. Bearing equality (4.17) in mind, we will represent series (4.15) with coefficients (4.16) in the form

$$\sum a_k(0)e^{-\omega_k t}X_k(x) = \sum a_k(0)X_k(x)\upsilon_k(t) + e^{-A_0^{-1}t}q_{10}(x)$$
$$\upsilon_k(t) = e^{-\omega_k t} - e^{-A_0^{-1}t}, \quad \lim_{k\to\infty}\omega_k = A_0^{-1} > 0. \tag{4.21}$$

The positive sequence $\{\upsilon_k(t)\}$ is non-increasing the the set $[0, T]$ and converges to zero uniformly in this set. In addition, as a consequence of the convergence of series (4.17) at the point x, the sequence $\{\sum_{k=1}^{n} a_k(0)X_k(x)\}$ is uniformly bounded in the set $[0, T]$. These properties enable the Dirichlet-Abel criterion [15] to be used and also enables us to establish that the series on the right-hand side of (4.21) and, consequently, series (4.15) also converges uniformly in $t \in [0, T]$.

After establishing that series (4.15) with coefficients (4.16) converges uniformly in $t \in [0, T]$, the correctness of equality (4.20) follows directly from the well-known theorem on the term-by-term integration of a uniformly converging series [15].

Assertion 2. Suppose we have the inclusion (4.18), the function $q_{10}(x)$ is odd and series (4.17) converges at each point of the segment $[-a, a]$ in the usual sense. Then the function $q_1(x, t)$ in the form of series (4.15) with coefficients (4.16) satisfies Eq. (4.11), it is odd with respect to x and assumption (4.4) holds for it.

Proof. We recall that expression (4.16) for the coefficients $a_k(t)$ of series (4.15) was obtained by substituting the latter into Eq. (4.11) and formally interchanging the order of the integration and summation operations. Having equalities (4.14) and (4.20) available, we can justify the correctness of similar rearrangements and thereby establish that series (4.15) with coefficients (4.16) satisfies Eq. (4.11).

To check the assumption that the function $q_1(x, t)$ is odd in x we will introduce the system of functions $Y_k(x) = X_k(-x)$, which correspond to the previous eigenvalues λ_k of the operator $\mathcal{L}$ and which also forms an orthonormalized basis in $L_2[-a, a]$. Using the fact that the function $q_{10}(x)$ is odd, we can establish that the expansion of the function $q_1(x, t)$ in the system $Y_k(x)$ differs from expansion (4.15) with coefficients (4.16) only in sign. Adding these expansions we obtain the equation

$$2q_1(x, t) = \sum a_k(0)e^{-\omega_k t}[X_k(x) - Y_k(x)]$$

which confirms that $q_1(x, t)$ is odd in x.

To check the assumption (4.4) we will use the Riesz–Fisher theorem [10], according to which, series (4.15) for a specified $t \in [0, T]$ converges in the root mean square to a certain function from $L_2[-a, a]$, provided the series $\sum a_k^2(t)$ converges for this t. The latter in fact occurs in view of relation (4.19) and inequality $a_k^2(t) \le a_k^2(0)$, which follows from relations (4.16).

Using Parseval's equality [12] for expansion (4.15) and taking expression (4.16) for $a_k(t)$ into account, we can write

$$\|q_1\|^2(t) = \sum a_k^2(t) = \sum a_k^2(0)e^{-2\omega_k t} \tag{4.22}$$

We recall that the sequence $\{\omega_k\}$ is defined by (4.16) in terms of positive constants A_0 and A_1 and is monotonically increasing. This fact, together with Eq. (4.19), enables us to obtain the following limit from relations (4.22)

$$\|q_1\|(t) \le \|q_{10}\|e^{-\omega_1 t}$$

which indicates the decay with time of the norm $\|q_1\|$. This property can be interpreted as the presence of relaxation with time of the function $q_1(x, t)$ from $L_2[-a, a]$.

Remark 2. Satisfaction of condition (4.18) ensures that series (4.17) converges in the root mean square, which does not guarantee the convergence of this series at points of the segment $[-a, a]$ [10], and hence the condition for term-by-term convergence of series (4.17), present in Assertions 1 and 2, has an independent character. On the other hand, for any specified function $q_{10}(x) \in L_2[-a, a]$ a function $q_{10}^*(x) \in L_2[-a, a]$ can be found which differs by as little as desired in the norm of the space $L_2[-a, a]$ from $q_{10}(x)$ and for which the condition for term-by-term convergence of series (4.17) in the segment $[-a, a]$ is satisfied; by virtue of the convergence of this series in the root mean square the following linear combination possesses this property

$$q_{10}^*(x) = a_1(0)X_1(x) + \ldots + a_{n_*}(0)X_{n_*}(x)$$

for sufficiently large integer $n_* > 0$.

Remark 3. We will put $n = 0$ in relations (4.15) and (4.16), which corresponds to the case of the absolutely rigid half-space considered in the previous section – expression (3.8) for $f = 0$. According to Eq. (4.16) the coefficients ω_k for $n = 0$ take the value $\omega_* = GV(\tau_a h)^{-1}$, which is independent of k, and hence, taking Eq. (4.17) into account, expansion (4.15) gives the expression

$$q_1(x, t) = q_{10}(x)e^{-\omega_* t} \tag{4.23}$$

which is identical with expression (3.8) mentioned above when $f = 0$.

In addition, we mention the analogy between expression (4.23) and expression (1.3) for the force T of elasticity of the spring tangential to the sliding surface in the problem in Section 1. In both cases the exponents ω and ω_* of the exponential decay of T and q_1 turn out to be directly proportional to the velocity V and the stiffness of the elastic system (γ and G/h) and inversely proportional to the frictional interaction parameter (F and τ_a).

Remark 4. A comparison of the formulations of the problems considered enables us to reveal the following fundamental conditions of the asymptotic form $q_1 \to 0$ as $t \to \infty$: the unidirectionality of the sliding of the punch (i.e. the fact that there is no displacement of the punch with respect to the system of coordinates in the direction of the x axis after sliding begins along the z axis), the fact that the contact area remains unchanged and the fact that the load on the punch along the y axis remains constant.

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